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Fock-Type Representation of the Lie Superalgebra $A(0, 1)$

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A Fock space of two pairs of generalized creation and annihilation operators is constructed. These operators belong to the odd part of the Lie superalgebra $A(0, 1)$ and generate the whole algebra. The creation and annihilation operators define in the Fock space an infinite-dimensional irreducible representation of the algebra $A(0, 1)$.

In the present note we study one particular infinite-dimensional representation of the Lie superalgebra $A(0, 1)$ in the Kac notation (Kac, 1977). The method we use is similar to the one applied in the quantum theory of bosons and fermions. For instance, n pairs of Fermi operators generate the Lie algebra B_n of the group $SO(2n + 1)$. Therefore the Fock space of these operators determines an irreducible representation of B_n. In a similar way the Fock space of Bose or, more generally, of para-Bose operators defines a class of infinite-dimensional representations of the orthosymplectic Lie superalgebra (Gantchev and Palev, 1978). The operators we introduce are neither Bose nor Fermi operators. Their representation space, however, possesses all main features of the ordinary Fock space. In fact it is generated out of a vacuum vector by means of polynomials of creation operators. We were led to these operators in a search for some possible generalizations of the quantum statistics. The present paper is an investigation along this line. It should not be considered as an attempt to develop a representation theory for the Lie superalgebras. Our main purpose is to study the Fock space of the operators we introduce by the simplest available example, so that later

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on it will be possible to generalize the results to the case of several and even an infinite number of creation and annihilation operators.

The relations between the generators of the algebra $A(0, 1)$ can be derived through its three-dimensional exact representation. Denote as $e_{\alpha\beta}$, $\alpha, \beta = -1, 0, 1$, a 3×3 matrix with 1 on the intersection of the αth row and βth column and zero elsewhere. Let L_0 and L_1 be subspaces of $A(0, 1)$ with the basis written in the brackets, namely,

$$\begin{aligned} L_0 &= \text{lin. env. } \{e_{-1,-1} + e_{00}, e_{00} + e_{11}, e_{1,-1}, e_{-1,1}\} \\ L_1 &= \text{lin. env. } \{e_{01}, e_{10}, e_{0,-1}, e_{-10}\} \end{aligned} \tag{1}$$

The multiplication $[\![\ ,\ ]\!]$ in $A(0, 1)$ is defined as follows:

$$\begin{aligned} [\![a, b]\!] &= \{a, b\} \equiv ab + ba, \qquad a, b \in L_1 \\ [\![a, b]\!] &= [a, b] \equiv ab - ba, \qquad a \text{ or } b \in L_0 \end{aligned} \tag{2}$$

and it is extended by linearity to the other elements.

In this case

$$A(0, 1) = L_0 + L_1 \tag{3}$$

and L_0, L_1 are the even and odd part of $A(0, 1)$, respectively.

The representation-independent structure relations of the generators can be derived from (2) and the multiplicative law of the matrices $e_{\alpha\beta}$,

$$e_{\alpha\beta} e_{\gamma\delta} = \delta_{\beta\gamma} e_{\alpha\delta} \tag{4}$$

Define the operators

$$\begin{aligned} A_1^{+} &= e_{10}, \qquad A_{-1}^{+} = -e_{0,-1} \\ A_1^{-} &= e_{01}, \qquad A_{-1}^{-} = e_{-1,0} \end{aligned} \tag{5}$$

These operators constitute a basis in L_1 and generate the whole algebra. Indeed, using (4) we obtain

$$\begin{aligned} \{A_1^{+}, A_1^{-}\} &= e_{11} + e_{00}, \qquad \{A_1^{+}, A_{-1}^{+}\} = -e_{1,-1} \\ \{A_{-1}^{+}, A_{-1}^{-}\} &= -e_{00} - e_{-1,-1}, \qquad \{A_1^{-}, A_{-1}^{-}\} = e_{-1,1} \end{aligned} \tag{6}$$

Let now a_η^{ξ}, $\xi, \eta = \pm$ or ± 1, be the representation-independent generators of the Lie superalgebra $A(0, 1)$, corresponding to A_η^{ξ}. Using the equality (4) we find the following structure relations between the operators a_η^{ξ}:[2]

$$\begin{aligned} [\{a_\xi^{\xi}, a_\eta^{-\eta}\}, a_\epsilon^{\epsilon}] &= \eta\delta_{\eta\epsilon} a_\xi^{\xi} - \eta\delta_{\xi\eta} a_\epsilon^{\epsilon} \\ [\{a_\xi^{\xi}, a_\eta^{-\eta}\}, a_\epsilon^{-\epsilon}] &= -\epsilon\delta_{\xi\epsilon} a_\eta^{-\eta} + \eta\delta_{\xi\eta} a_\epsilon^{-\epsilon} \\ \{a_\xi^{\xi}, a_\eta^{\eta}\} &= \{a_{-\xi}^{\xi}, a_{-\eta}^{\eta}\} = 0 \end{aligned} \tag{7}$$

In this notation

$$\begin{aligned} L_1 &= \text{lin. env. } \{a_\eta^{\xi} \mid \xi, \eta = \pm\} \\ L_0 &= \text{lin. env. } \{\{a_\xi^{\xi}, a_\eta^{-\eta}\} \mid \xi, \eta = \pm\} \end{aligned} \tag{8}$$

[2] Throughout the paper $\xi, \eta, \epsilon = \pm$ or ± 1; $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$.

Definition. We call the operators a_η^ξ creation ($\xi = +$) and annihilation ($\xi = -$) operators.

By representation of the creation and annihilation operators we understand a mapping

$$\theta: a_\eta^\xi \to \tilde{a}_\eta^\xi \tag{9}$$

of the operators a_η^ξ onto a set of linear operators $\tilde{a}_\eta^\xi$, that preserves the relations (7). Since the creation and annihilation operators generate the Lie superalgebra $A(0, 1)$, to every representation of the operators a_η^ξ there corresponds through (8) a representation of $A(0, 1)$, and vice versa. Moreover both representations are simultaneously reducible or irreducible. Thus the problem of finding the representations of the algebra $A(0, 1)$ reduces completely to the problem of finding all representations of the creation and annihilation operators.

Let W be the representation space we are looking for. We assume that the space contains a vector $|0\rangle \in W$ called a vacuum such that

$$a_\eta^-|0\rangle = 0, \qquad \eta = \pm \tag{10}$$

In order to obtain a space generated out of the vacuum by means of the creation operators we postulate that

$$\begin{aligned} a_1^- a_1^+|0\rangle &= p|0\rangle \\ a_{-1}^- a_{-1}^+|0\rangle &= q|0\rangle \end{aligned} \tag{11}$$

This requirement is a natural generalization of the equation

$$a_i^- a_j^+|0\rangle = \delta_{ij} p|0\rangle \tag{12}$$

used in the parastatistics (Green, 1953) in order to single out an irreducible Fock space. In our case

$$\{a_1^-, a_{-1}^+\} = \{a_{-1}^-, a_1^+\} = 0 \tag{13}$$

so that the equations (11) are enough.

The scalar product in W is determined in the usual way:

(a) $(a_{\eta_1}^+ a_{\eta_2}^+ \cdots a_{\eta_m}^+|0\rangle, a_{\xi_1}^+ \cdots a_{\xi_n}^+|0\rangle) = \langle 0|a_{\eta_m}^- \cdots a_{\eta_2}^- a_{\eta_1}^- a_{\xi_1}^+ \cdots a_{\xi_n}^+|0\rangle$ (14)
(b) $\langle 0|a_\eta^+ = 0$
(c) $\langle 0|0\rangle = 0$

It is not clear from the beginning whether the definition (a)–(c) together with (7) and (11) gives a metric in W. In fact, this is not the case for arbitrary p and q (for instance $p = -q = 1$). The requirement

$$(a, a) > 0 \qquad \forall 0 \neq a \in W \tag{15}$$

appears as an additional restriction on the constants p and q.

In this paper we shall consider the simplest nontrivial case $p = 1$, $q = 0$, i.e., we require[3]

$$a_1^-|0\rangle = a_{-1}^-|0\rangle = a_{-1}^{\pm}|0\rangle = 0 \tag{16}$$

Lemma 1. The representation space W is a linear envelope of all vectors

$$\{a_{-1}^{\pm}, a_1^+\}^n a_1^+|0\rangle, \qquad \{a_{-1}^{\pm}, a_1^+\}^n|0\rangle, \qquad n = 0, 1, 2, \ldots \tag{17}$$

Proof. The representation space W is spanned on all possible vectors

$$a_{\eta_1}^{\xi_1} a_{\eta_2}^{\xi_2} \cdots a_{\eta_m}^{\xi_m}|0\rangle, \qquad m = 0, 1, 2, \ldots; \; \xi_i, \eta_i = \pm \tag{18}$$

To prove the lemma, we have to show that every vector (19) is a linear combination of the vectors (17). For this purpose we shall use the Poincaré–Birkhoff–Witt theorem (Milnor, 1965), which in our case can be formulated in the following way. Let $L = L_0 + L_1$ be a Lie superalgebra, $a_1, \ldots, a_m$ be a basis in L_0, and $b_1, b_2, \ldots, b_n$ a basis in L_1. Then the elements

$$a_1^{K_1} b_1^{\theta_1} a_2^{K_2} \cdots a_m^{K_m} b_2^{\theta_2} \cdots b_n^{\theta_n}, \qquad K_i \geqslant 0, \qquad \theta_i = 0, 1 \tag{19}$$

form a basis in the universal enveloping algebra of L.

For the Lie superalgebra $A(0, 1)$ the theorem gives that the monomials

$$(n, \theta, m_1, m_2, m_3, \theta_1, \theta_2, \theta_3)$$
$$\equiv \{a_{-1}^{\pm}, a_1^+\}^n (a_1^+)^{\theta} \{a_1^-, a_1^+\}^{m_1} \{a_{-1}^-, a_{-1}^{\pm}\}^{m_2} \{a_{-1}^-, a_1^-\}^{m_3} (a_{-1}^{\pm})^{\theta_1} (a_{-1}^-)^{\theta_2} (a_1^-)^{\theta_3} \tag{20}$$

define a basis in the universal enveloping algebra of $A(0, 1)$. Hence the monomial

$$a_{\eta_1}^{\xi_1}, a_{\eta_2}^{\xi_2} \cdots a_{\eta_m}^{\xi_m} \tag{21}$$

is a linear combination of vectors $(n, \theta, \ldots, \theta_3)$. Therefore

$$a_{\eta_1}^{\xi_1} a_{\eta_2}^{\xi_2} \cdots a_{\eta_m}^{\xi_m}|0\rangle = \sum_{n, \cdots, \theta_3} \alpha_{n \cdots \theta_3}(n, \theta, m_1, m_2, m_3, \theta_1, \theta_2, \theta_3)|0\rangle \tag{22}$$

where $\alpha_{n \cdots \theta_3}$ are number coefficients.

Since

$$(n, \theta, m_1, \ldots, \theta_3)|0\rangle \neq 0 \qquad \text{only if } m_2 = m_3 = \theta_1 = \theta_2 = \theta_3 = 0 \tag{23}$$

and

$$(n, \theta, m_1, 0, \ldots, 0)|0\rangle = \{a_{-1}^{\pm}, a_1^+\}^n (a_1^+)^{\theta}|0\rangle \tag{24}$$

we conclude

[3] A similar representation for the case of several creation and annihilation operators generating the Lie algebra A_n was studied in Palev (1978).

$$a_{\eta_1}^{\xi_1} \cdots a_{\eta_n}^{\xi_n}|0\rangle = \sum_{n,\theta} \alpha_{n,\theta}\{a_{-1}^{\pm}, a_1^{+}\}^n (a_1^{+})^{\theta}|0\rangle \tag{25}$$

This completes the proof.

We now proceed to find the transformation properties of the vectors (17) under the left multiplications with creation and annihilation operators. It is convenient to represent the space as a direct sum of its even and odd subspaces, W_0 and W_1, respectively,

$$W = W_0 + W_1 \tag{26}$$

where

$$\begin{aligned} W_0 &= \text{lin. env. } \{\{a_{-1}^{\pm}, a_1^{+}\}^n|0\rangle | n = 0, 1, 2, \ldots\} \\ W_1 &= \text{lin. env. } \{\{a_{-1}^{\pm}, a_1^{+}\}^n a_1^{+}|0\rangle | n = 0, 1, 2, \ldots\} \end{aligned} \tag{27}$$

Denote as

$$|n, \theta\rangle = \{a_{-1}^{\pm}, a_1^{+}\}^n (a_1^{+})^{\theta}|0\rangle, \qquad \theta = 0, 1;\ n = 0, 1, 2, \ldots \tag{28}$$

From the structure relations (7) we have

$$[\{a_{-1}^{\pm}, a_1^{+}\}^n, a_\eta^{+}] = 0 \tag{29}$$

Therefore

$$a_\eta^{+}|n, 0\rangle = \{a_{-1}^{\pm}, a_1^{+}\}^n a_\eta^{+}|0\rangle \tag{30}$$

and taking into account (16) we obtain

$$a_1^{+}|n, 0\rangle = |n, 1\rangle, \qquad a_{-1}^{\pm}|n, 0\rangle = 0 \tag{31}$$

Since $a_\eta^{+} a_\eta^{+} = 0$,

$$a_1^{+}|n, 1\rangle = 0 \tag{32}$$

For $a_{-1}^{\pm}$ we have

$$a_{-1}^{\pm}|n, 1\rangle = \{a_{-1}^{\pm}, a_1^{+}\}^n a_{-1}^{\pm} a_1^{+}|0\rangle = |n + 1, 0\rangle \tag{33}$$

To calculate the transformation properties with respect to the annihilation operators, we use the identity

$$[a_\eta^{-}, \{a_{-1}^{\pm}, a_1^{+}\}^n] = n\{a_{-1}^{\pm}, a_1^{+}\}^{n-1} a_{-\eta}^{\pm} \tag{34}$$

We have

$$a_\eta^{-}|n, 0\rangle = [a_\eta^{-}, \{a_{-1}^{\pm}, a_1^{+}\}^n|0\rangle = n\{a_{-1}^{\pm}, a_1^{+}\}^{n-1} a_{-\eta}^{\pm}|0\rangle \tag{35}$$

Therefore

$$a_1^{-}|n, 0\rangle = 0, \qquad a_{-1}^{\mp}|n, 0\rangle = n|n - 1, 1\rangle \tag{36}$$

Similarly

$$\begin{aligned} a_{-1}^{\mp}|n, 1\rangle &= [a_{-1}^{\mp}, \{a_{-1}^{\pm}, a_1^{+}\}^n] a_1^{+}|0\rangle = n\{a_{-1}^{\pm}, a_1^{+}\}^n (a_1^{+})^2|0\rangle = 0 \\ a_1^{-}|n, 1\rangle &= [a_1^{-}, \{a_{-1}^{\pm}, a_1^{+}\}^n] a_1^{+}|0\rangle + \{a_{-1}^{\pm}, a_1^{+}\}^n a_1^{-} a_1^{+}|0\rangle \\ &= (n + 1)|n, 0\rangle \end{aligned} \tag{37}$$

We summarize the results

$$
\begin{aligned}
&a_{-1}^{+}W_0 = a_1^{-}W_0 = a_1^{+}W_1 = a_{-1}^{-}W_1 = 0 \\
&a_1^{+}|n, 0\rangle = |n, 1\rangle, \qquad a_{-1}^{+}|n, 1\rangle = |n+1, 0\rangle \\
&a_{-1}^{-}|n, 0\rangle = n|n-1, 1\rangle, \qquad a_1^{-}|n, 1\rangle = (n+1)|n, 0\rangle
\end{aligned} \tag{38}
$$

We are now ready to calculate explicitly the scalar product in W.

Lemma 2. The vectors

$$|n, \theta\rangle, \qquad n = 0, 1, 2, \ldots; \theta = 0, 1 \tag{39}$$

define an orthogonal basis in the representation space W.

Proof. We make use of the following relations,

$$(a_1^{-})^{\theta_1}|n, \theta_2\rangle = (1 - \theta_1)|n, \theta_2\rangle + \theta_1\theta_2(n+1)|n, 0\rangle \tag{40}$$

$$\{a_{-1}^{-}, a_1^{-}\}^m|n, \theta\rangle = \frac{n!}{(n-m)!}\frac{(n+\theta)!}{(n+\theta-m)!}|n-m, \theta\rangle, \qquad m \leqslant n \tag{41}$$

$$\{a_{-1}^{-}, a_1^{-}\}^m|n, \theta\rangle = 0, \qquad m > n \tag{42}$$

The first equality is an immediate consequence of (38). The second one can be proved by induction. From (38) we have

$$\{a_{-1}^{-}, a_1^{-}\}|n, \theta\rangle = n(n+\theta)|n-1, \theta\rangle \tag{43}$$

Suppose (41) holds. For $m + 1 \leqslant n$ we have

$$
\begin{aligned}
\{a_{-1}^{+}, a_1^{-}\}^{m+1}|n, \theta\rangle &= \frac{n!}{(n-m)!}\frac{(n+\theta)!}{(n+\theta-m)!}\{a_{-1}^{-}, a_1^{-}\}|n-m, \theta\rangle \\
&= \frac{n!}{[n-(m+1)]!}\frac{(n+\theta)!}{[n+\theta-(m+1)]!}|n-(m+1), \theta\rangle
\end{aligned} \tag{44}
$$

i.e., for $m + 1$ the formula (41) also holds. The relation (42) is evident since

$$\{a_{-1}^{-}, a_1^{-}\}^{n+1}|n, \theta\rangle = n!\,(n+\theta)!\,\{a_{-1}^{-}, a_1^{-}\}|0, \theta\rangle = 0 \tag{45}$$

Using the definition (14) we calculate the scalar product between the vectors $|m, \theta_1\rangle$ and $|n, \theta_2\rangle$:

$$S \equiv (|m, \theta_1\rangle, |n, \theta_2\rangle) = \langle 0|(a_1^{-})^{\theta_1}\{a_{-1}^{-}, a_1^{-}\}^m|n, \theta_2\rangle \tag{46}$$

If $m > n$ according to (42) $S = 0$. Let $m \leqslant n$. Using first (41) and then (40) we obtain

$$
\begin{aligned}
S = {}& \frac{n!\,(n+\theta_2)!}{(n-m)!\,(n+\theta_2-m)!} \\
&\times \{(1-\theta_1)\langle 0|n-m, \theta_2\rangle + \theta_1\theta_2(n-m+1)\langle 0|n-m, 0\rangle\}
\end{aligned} \tag{47}
$$

If $m < n$ then S vanishes since

$$\langle 0|\{a^{\pm}_{-1}, a_1{}^{+}\}^{n-m} = 0 \tag{48}$$

For $m = n$ the expression in the brackets of (47) is nonzero only for $\theta_1 = \theta_2$. Therefore we obtain

$$(|m, \theta_1\rangle, |n, \theta_2\rangle) = \delta_{nm}\delta_{\theta_1\theta_2} n!\,(n + \theta_2)! \tag{49}$$

This proves the lemma.

The orthonormal basis is

$$|n, \theta) = \frac{\{a_1{}^{+}, a^{+}_{-1}\}^n (a_1{}^{+})^{\theta}}{\sqrt{n!\,(n + \theta)!}} |0\rangle, \qquad n = 0, 1, 2, \ldots, \qquad \theta = 0, 1 \tag{50}$$

In terms of this basis we have

$$\begin{gathered} a^{+}_{-1}W_0 = a_1{}^{-}W_0 = a_1{}^{+}W_1 = a^{-}_{-1}W_1 = 0 \\ a_1{}^{+}|n, 0) = (n + 1)^{1/2}|n, 1) \qquad a^{+}_{-1}|n, 1) = (n + 1)^{1/2}|n + 1, 0) \\ a^{-}_{-1}|n, 0) = n^{1/2}|n - 1, 1) \qquad a_1{}^{-}|n, 1) = (n + 1)^{1/2}|n, 0) \end{gathered} \tag{51}$$

The formulas (51) determine an infinite-dimensional representation of the Lie superalgebra $A(0, 1)$. In the metric (49)

$$(a_\eta{}^{+})^* = a_\eta{}^{-} \tag{52}$$

where the asterisk means Hermitian conjugation. The matrix elements of the even generators can be easily calculated from (51) taking into account (8).

In this paper we have not tried to ascribe a physical meaning to the creation and annihilation operators. Remark, however, that in the "particle" terminology the vector $|n, \theta\rangle$ corresponds to the $(2n + \theta)$-particle state since it is obtained from the vacuum by means of a homogeneous polynomial of order $2n + \theta$. The operator

$$H = \{a_1{}^{+}, a_1{}^{-}\} + \{a^{+}_{-1}, a^{-}_{-1}\} - 1 \tag{53}$$

has the properties of a free Hamiltonian. The spectrum of H is positive definite

$$H|n, \theta\rangle = (2n + \theta)|n, \theta\rangle \tag{54}$$

Moreover

$$[H, a_\eta{}^{\pm}] = \pm a_\eta{}^{\pm} \tag{55}$$

Therefore if $|E\rangle$ is a state with energy E and $a_\eta{}^{\xi}|E\rangle \neq 0$, then $a_\eta{}^{+}$ (or, respectively, $a_\eta{}^{-}$) increases (decreases) the energy by 1. Hence $a_\eta{}^{\xi}$ can be interpreted as an operator creating (annihilating) a particle of sort η.

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